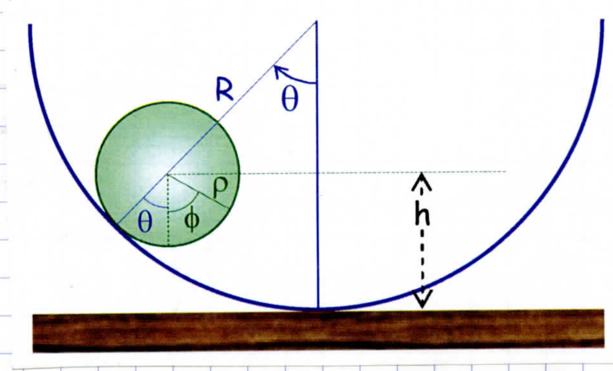
TM5 Pr 7.3

A SPHERE OF RADIUS ρ IS CONSTRAINED TO ROLL INSIDE HOLLOW CYLINDER OF RADIUS R. DETERMINE THE LAGRANGIAN FUNCTION, THE EQUATION OF CONSTRAINT AND SHOW



$$-(R-\rho)mg\sin\theta - m(R-\rho)^2\ddot{\theta} + \lambda(R-\rho) = 0$$

$$-\tfrac{2}{5}m\rho^2\ddot{\phi} - \lambda\rho = 0$$

WHERE $\underbrace{\lambda = -\tfrac{2}{5}m(R-\rho)\ddot{\theta} = -\tfrac{2}{5}m\rho\ddot{\phi}}_{M}$ AND $\omega_N = \sqrt{\dfrac{5g}{7(R-\rho)}}$

WRITE THE POSITION OF THE SPHERE RELATIVE TO THE BOTTOM

$$y = R - (R-\rho)\cos\theta$$

WRITE THE ENERGIES

$$T = \tfrac{1}{2}m\dot{s}^2 + \tfrac{1}{2}I\dot{\phi}^2 = \tfrac{1}{2}m(R-\rho)^2\dot{\theta}^2 + \tfrac{1}{2}\left(\tfrac{2}{5}m\rho^2\right)\dot{\phi}^2$$

$$\underline{T = \tfrac{1}{2}m(R-\rho)^2\dot{\theta}^2 + \tfrac{1}{5}m\rho^2\dot{\phi}^2}$$

$$U = mgh = \underline{mg[R-(R-\rho)\cos\theta] = U}$$

WRITE THE CONSTRAINT MOTION OF CM = DIST. ROLLED

$$s = r\phi \Rightarrow (R-\rho)\theta = \rho\phi$$

$$\Rightarrow \underline{f = (R-\rho)\theta - \rho\phi = 0}$$

THE LAGRANGIAN IS

$$L = \tfrac{1}{2}m(R-\rho)^2\dot{\theta}^2 + \tfrac{1}{2}m\rho^2\dot{\phi}^2 - mg[R-(R-\rho)\cos\theta]$$

APPLY LAGRANGE'S EQUATIONS WITH UNDETERMINED MULTIPLIERS

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \sum_k \lambda_k \frac{\partial f}{\partial q_j} = 0$$

$\longrightarrow$

$$\frac{\partial L}{\partial \theta} = -mg(R-\rho)\sin\theta$$

$$\frac{\partial L}{\partial \dot\theta} = m(R-\rho)^2\dot\theta$$

$$\frac{\partial f}{\partial \theta} = (R-\rho)$$

$$\boxed{-mg(R-\rho)\sin\theta - m(R-\rho)^2\ddot\theta + \lambda(R-\rho) = 0}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot\phi} = \frac{2}{5}m\rho^2\dot\phi$$

$$\frac{\partial f}{\partial \phi} = -\rho$$

$$\boxed{-\frac{2}{5}m\rho^2\ddot\phi - \lambda\rho = 0}$$

$$\Rightarrow \boxed{\lambda = -\frac{2}{5}m\rho\ddot\phi} \quad \text{QED}$$

USING THE CONSTRAINT THAT $(R-\rho)\ddot\theta = \rho\ddot\phi$

$$\Rightarrow \boxed{\lambda = -\frac{2}{5}m(R-\rho)\ddot\theta} \quad \text{QED}$$

THE FREQUENCY OF SMALL OSCILLATIONS, $\dot\theta$, CAN BE FOUND FROM LAGRANGE'S EQUATION IN θ:

$$\cancel{m}(R-\rho)^{\cancel{2}}\ddot\theta + \cancel{m}g\cancel{(R-\rho)}\sin\theta + \cancel{(R-\rho)}\left[-\frac{2}{5}\cancel{m}(R-\rho)\ddot\theta\right] = 0$$

$$\left(1+\frac{2}{5}\right)(R-\rho)\ddot\theta + g\sin\theta = 0$$

FOR SMALL OSCILLATIONS, $\sin\theta \rightarrow \theta$

$$\frac{7}{5}(R-\rho)\ddot\theta + g\theta = 0$$

$$\Rightarrow \quad \ddot\theta + \left[\frac{5g}{7(R-\rho)}\right]\theta = 0$$

$$\rightarrow \boxed{\omega_N = \sqrt{\frac{5g}{7(R-\rho)}}} \quad \text{QED!}$$